FACTORIZATION OF ALTERNATING SUMS OF VIRASORO CHARACTERS

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Abstract. G. Andrews proved that if n is a prime number then the coefficients a_k and a_{k+n} of the product $(q,q)_\infty/(q^n,q^n)_\infty = \sum_k a_k q^k$ have the same sign, see [A1]. We generalize this result in several directions. Our results are based on the observation that many products can be written as alternating sums of characters of Virasoro modules.

1. Introduction

In the past several decades with the appearance and rapid development of Conformal Field Theory, the Virasoro modules enjoyed ample attention from mathematicians and physicists.

In this paper we study the following question.

Question. Which finite (alternating) sum of characters of Virasoro modules occurring in a minimal series can be written in the form

$$\frac{\prod_{i\in I_-}(1-q^i)\ \prod_{i\in I_+}(1+q^i)}{\prod_{j\in J_-}(1-q^j)\ \prod_{j\in J_+}(1+q^j)}\ ,$$

where $I_\pm, J_\pm$ are some sets of natural numbers?

This question is motivated by a number of applications to combinatorics and mathematical physics.

- The character of any single Virasoro module occurring in $(2,2r+1)$ minimal series is factorizable. The same character can be written in a so-called fermionic form and we obtain the celebrated Rogers-Ramanujan-Gordon-Andrews identities, see [A2].
- For any $1<s<p'$, the sums and differences of characters of $(1,s)$ and $(p-1,s)$ modules in (p,p') minimal series, where $p\in\{3,4\}$, are factorizable. For the case of the sum we also have a fermionic formula and therefore an identity of Rogers-Ramanujan-Gordon-Andrews type, see [FFW].
- A sum of three Virasoro characters from $(2,9)$ minimal series equals the product $(q,q)_\infty/(q^3,q^3)_\infty$. It immediately implies that if $(q,q)_\infty/(q^3,q^3)_\infty=\sum_k a_k q^k$ and if a_k and a_{k+3} are non zero then they have the same sign. The famous Borwein conjecture is a finitization of this fact: it asserts that if coefficients $a_k^{(N)}$ and $a_{k+3}^{(N)}$

of the product $\prod_{j=0}^{N}(1-q^{3j+1})(1-q^{3j+2}) = \sum_k a_k^{(N)} q^k$ are non-zero then they have the same sign, see [A1].

- The number 1 can be written in several ways as an alternating sum of characters of Virasoro modules from $(2, 2r+1)$ minimal series. Each such equality gives a family of non-trivial partition identities, see [MMO].
- Factorized form of graded characters of Virasoro modules is crucial for studying form factors of integrable deformations of Conformal Field Theory, see [Ch].
- Writing a sum of graded characters of Virasoro modules in a product form leads to identities involving sums of products of graded characters of Virasoro modules.

We present two large families of products which are equal to finite alternating sums of Virasoro characters. In particular, these families contain all the known examples of such phenomena.

We prove our formulae by an application of the triple Jacobi identity and of the quintuple identity to the Rocha-Caridi formula for the characters of Virasoro modules.

In the case of the triple Jacobi identity we prove the following formula

$$\frac{(q^{\frac{B(a'-c)}{2}}, q^{\frac{B(a'+c)}{2}}, q^{Ba'}; q^{Ba'})_\infty}{(q^n;q^n)_\infty} = q^{\frac{(p-p')^2-(cB)^2}{8Ba'}} \sum_{\substack{0<r<p/b,\ r\equiv 1 \pmod 2,\\ 0<s<p'/b',\ ps\equiv bc \pmod{a'}}} (-1)^{t_{r,s}} \chi^{(p,p')}_{rb,sb'}(q^n)\ , \tag{1.1}$$

where

$$t_{r,s} = \frac{p'r/b' - ps/b + c}{2}\ .$$

Here p, p' are two relatively prime positive integers, and a', b, b', c are positive integers such that $2b$ divides p, $a'b'$ divides p', $a' > c$ and c is odd. The numbers B and n are given by $B = bb'$, $n = \frac{pp'}{2a'bb'}$. The right hand side of our formula contains an alternating sum of n different Virasoro characters from (p, p') minimal series.

A few cases of such formula are known. The cases of $n = 1, 2$ can be found in [BF], [FFW]. The case of $b = b' = B = 1$, $a' = 3n$, $n = c$ can be found in [MMO] (in this case the left hand side clearly equals to 1).

If n is even (that is if p is divisible by 4), the signs in the formula can be written in a different way. Namely, for the case of even n we also have:

$$\frac{(q^{\frac{B(a'-c)}{2}}, -q^{\frac{B(a'+c)}{2}}, -q^{Ba'}; -q^{Ba'})_\infty}{(q^n;q^n)_\infty} = q^{\frac{(p-p')^2-(cB)^2}{8Ba'}} \sum_{\substack{0<r<p/b,\ r\equiv 1 \pmod 2,\\ 0<s<p'/b',\ ps\equiv bc \pmod{a'}}} (-1)^{\frac{t_{r,s}(t_{r,s}+1)}{2}} \chi^{(p,p')}_{rb,sb'}(q^n)\ .$$

if $a' - c$ is divisible by 4, and

$$\frac{(-q^{\frac{B(a'-c)}{2}}, q^{\frac{B(a'+c)}{2}}, -q^{Ba'}; -q^{Ba'})_\infty}{(q^n;q^n)_\infty} = q^{\frac{(p-p')^2-(cB)^2}{8Ba'}} \sum_{\substack{0<r<p/b,\ r\equiv 1 \pmod 2,\\ 0<s<p'/b',\ ps\equiv bc \pmod{a'}}} (-1)^{\frac{t_{r,s}(t_{r,s}-1)}{2}} \chi^{(p,p')}_{rb,sb'}(q^n)\ .$$

if $a' - c$ is not divisible by 4. Some cases of these formulae with $n = 2$ are contained in [BF], [FFW].

The above formulae can be used to obtain identities of Rogers-Ramanujan-Gordon-Andrews type by equating the product side with any known expression for Virasoro characters in the right hand side. In particular, we expect that the known fermionic expressions for the Virasoro characters appearing in our formula (see e.g. [BM], [BMS], [W]) can be summed up to a fermionic form. We do not discuss fermionic formulae in this paper.

Another set of identities is obtained by multiplying or dividing the product forms for different cases of the above formula. Such identities involve sums of products of Virasoro characters, see Remark 3.15.

We use our formulae to study signs of coefficients of products. Write the left hand side of (1.1) as a formal power series $\sum_{j=0}^{\infty} \phi_j q^j$. We conjecture that ϕ_j and ϕ_{j+n} always have the same sign and prove it in several cases, see Theorem 3.5. The case $a' = 3$, $B = c = 1$ and prime n was proved in [A1] in relation to the Borwein conjecture.

In some cases (e.g. when $B = 1$ and all odd prime divisors of n divide a') for each j there is only one term on the right hand side of (1.1) which has a non-trivial coefficient of q^j, and it follows that ϕ_j and ϕ_{j+n} do have the same sign. In more complicated cases, one can hope to make use of some fermionic expressions for Virasoro characters to perform the subtraction. We use this idea to prove our conjecture for the case of prime n and odd B.

The formulae and results in the case of the quintuple identity are similar, see formulae (3.4), (3.5), (3.6), (3.7) and Theorem 3.12.

Our paper is structured as follows. We recall basic facts about Virasoro modules in Section 2. Sections 3.1 and 3.2 contain statements of the main results in the cases of the triple and quintuple products respectively. The proofs are collected in Section 4.

2. Minimal models

Let $\mathcal{V}ir$ be the Virasoro algebra with the standard $\mathbb{C}$-basis $\{L_n\}_{n\in\mathbb{Z}}$ and C, satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}m(m^2-1)\delta_{m+n,0}, \quad [C, L_n] = 0.$$

Let (p, p') be a pair of relatively prime integers greater than 1. There exists a family of irreducible $\mathcal{V}ir$-modules $M_{r,s}^{(p,p')}$ where $1 \le r \le p-1$, $1 \le s \le p'-1$ on which C acts as the scalar

$$C_{p,p'} = 1 - \frac{6(p-p')^2}{4pp'} \ .$$

The module $M_{r,s}^{(p,p')}$ is $\mathbb{Q}$-graded with respect to the degree operator L_0 and the corresponding formal character $\chi_{r,s}^{(p,p')}(q) := \mathrm{Tr}(q^{L_0})$ is given by the following bosonic formula,

see [RC]:

$$\chi_{r,s}^{(p,p')}(q) := \frac{q^{\Delta_{r,s}^{(p,p')}}}{(q)_\infty}\left(\sum_{j\in\mathbb{Z}} q^{pp'j^2+(p'r-ps)j} - \sum_{j\in\mathbb{Z}} q^{pp'j^2+(p'r+ps)j+rs}\right).$$

Here $(q)_\infty = \prod_{j=1}^\infty (1-q^j)$ and the conformal dimension $\Delta_{r,s}^{(p,p')}$ is given by

$$\Delta_{r,s} = \frac{(p'r-sp)^2-(p'-p)^2}{4pp'}\ .$$

It is convenient to write $\chi_{r,s}^{(p,p')}(q)$ in the following form:

$$\chi_{r,s}^{(p,p')}(q) = \frac{q^{-\frac{(p'-p)^2}{4pp'}}}{(q)_\infty}\left(\sum_{j\in\mathbb{Z}} q^{\frac{(2pp'j+p'r-ps)^2}{4pp'}} - \sum_{j\in\mathbb{Z}} q^{\frac{(2pp'j+p'r+ps)^2}{4pp'}}\right). \tag{2.1}$$

From formula (2.1) one immediately observes that

$$\chi_{r,s}^{(p,p')}(q) = \chi_{p-r,p'-s}^{(p,p')}(q). \tag{2.2}$$

The normalized character $\bar\chi_{r,s}^{(p,p')}$ given by

$$\bar\chi_{r,s}^{(p,p')}(q) := q^{-\Delta_{r,s}^{(p,p')}}\chi_{r,s}^{(p,p')}(q) = 1+o(1)$$

is a formal power series in q with non-negative coefficients. (In fact the only zero coefficient is the coefficient of q in $\bar\chi_{1,1}^{(p,p')} = \bar\chi_{p-1,p'-1}^{(p,p')}$.)

3. Main results

For integers a,b, we write $a \perp b$ if a,b are relatively prime.

Let p,p' be relatively prime integers greater than 1. Let a,a',b,b' be natural numbers and c a non-negative integer such that $a'>c$, ab divides p, $a'b'$ divides p'.

We call numbers b,b' *the scaling factors*, numbers a,a' *the moduli* and number c *the common residue*. We obviously have $ab \perp a'b'$.

We define

$$B := bb', \qquad n := \frac{pp'}{aa'bb'}.$$

We use the notation $(u_1,\dots,u_k;v)_\infty := \prod_{i=0}^\infty\prod_{j=1}^k (1-u_jv^i)$.

3.1. **Triple products.**

In this section we assume that c is odd and set

$$a=2.$$

Then p',c,a',b' are all odd, p/b is even.

We call a pair of integers (r,s) *2-contributing of the first type* if

$$0<r<p/b, \qquad 0<s<p'/b', \qquad \frac{p'r/b'-ps/b+c}{2aa'} \in \mathbb{Z}.$$

We call a pair of integers (r,s) *2-contributing of the second type* if

$$0<r<p/b, \qquad 0<s<p'/b', \qquad \frac{p'r/b'+ps/b-c}{2aa'}\in\mathbb{Z}.$$

We denote the set of all 2-contributing pairs of type j by $\mathcal{A}_j^{(2)}$, $j=1,2$. We call pair of integers (r,s) *2-contributing* if (r,s) is either 2-contributing of the first type or 2-contributing of the second type. We denote the set of all 2-contributing pairs by $\mathcal{A}^{(2)}$.

Lemma 3.1. *We have* $\mathcal{A}_1^{(2)}\bigcap\mathcal{A}_2^{(2)}=\emptyset$.
If $0<r<r+a<p/b$, *then* $(r,s)\in\mathcal{A}_j^{(2)}$ *if and only if* $(r+a,s)\in\mathcal{A}_{3-j}^{(2)}$.
If $0<s<s+a'<p'/b'$ *and* p *is even then* $(r,s)\in\mathcal{A}_j^{(2)}$ *if and only if* $(r,s+a')\in\mathcal{A}_j^{(2)}$.
If $0<s<s+a'<p'/b'$ *and* p *is odd then* $(r,s)\in\mathcal{A}_j^{(2)}$ *if and only if* $(r,s+a')\in\mathcal{A}_{3-j}^{(2)}$.
□

Theorem 3.2. *We have the following identity of formal power series in* q:

$$\frac{(q^{\frac{B(a'-c)}{2}},q^{\frac{B(a'+c)}{2}},q^{Ba'};q^{Ba'})_\infty}{(q^n;q^n)_\infty}=q^{\frac{(p-p')^2-(cB)^2}{4Baa'}}\Big(\sum_{(r,s)\in\mathcal{A}_1^{(2)}}\chi^{(p,p')}_{rb,sb'}(q^n)-\sum_{(r,s)\in\mathcal{A}_2^{(2)}}\chi^{(p,p')}_{rb,sb'}(q^n)\Big). \quad (3.1)$$

Theorem 3.2 is proved in Section 4.1.

The cases $n=1,2$ of Theorem 3.2 can be found in [BF], see also [FFW]. The case $b=b'=B=1$, $a'=3n$, $n=c$ of Theorem 3.2 can be found in [MMO].

We note that there are n summands on the left hand side of (3.1), moreover, $\chi^{(p,p')}_{rb,sb'}$ is present only if $\chi^{(p,p')}_{p-rb,p'-sb'}$ is not present, see Lemma 4.1. We also note that formula (3.1) remains the same if c is changed to $-c$, the right hand side for the obvious reason and the left hand side because of relation (2.2).

If n is even then there is a formula which differs from (3.1) only by the choice of signs.

For a 2-contributing pair (r,s) we define the integer $t_{r,s}$ by the formula

$$t_{r,s}=(p'r/b'-ps/b+c)/2.$$

The integer $t_{r,s}$ is even if $(r,s)\in\mathcal{A}_1^{(2)}$ and odd if $(r,s)\in\mathcal{A}_2^{(2)}$.

Theorem 3.3. *Let* n *be even and let* $a'\equiv c \pmod 4$. *Then we have the following identity of formal power series in* q:

$$\frac{(q^{\frac{B(a'-c)}{2}},-q^{\frac{B(a'+c)}{2}},-q^{Ba'};-q^{Ba'})_\infty}{(q^n;q^n)_\infty}=q^{\frac{(p-p')^2-(cB)^2}{4Baa'}}\Big(\sum_{(r,s)\in\mathcal{A}^{(2)}}(-1)^{\frac{t_{r,s}(t_{r,s}+1)}{2}}\chi^{(p,p')}_{rb,sb'}(q^n)\Big). \quad (3.2)$$

Let n *be even and let* $a'\not\equiv c \pmod 4$. *Then we have the following identity of formal power series in* q:

$$\frac{(-q^{\frac{B(a'-c)}{2}},q^{\frac{B(a'+c)}{2}},-q^{Ba'};-q^{Ba'})_\infty}{(q^n;q^n)_\infty}=q^{\frac{(p-p')^2-(cB)^2}{4Baa'}}\Big(\sum_{(r,s)\in\mathcal{A}^{(2)}}(-1)^{\frac{t_{r,s}(t_{r,s}-1)}{2}}\chi^{(p,p')}_{rb,sb'}(q^n)\Big). \quad (3.3)$$

Theorem 3.3 is proved in Section 4.2.

Some cases with $n = 2$ of Theorem 3.3 can be found in [BF], see also [FFW].

We apply Theorem 3.2 to study the signs of the coefficients of products.

Fix natural numbers a', B, c, n such that $a' > c$, $a'c \perp 2$. Define formal power series $\phi(q)$ by the formula:

$$\phi(q) = \phi_{a',B,c,n}(q) := \frac{(q^{\frac{B(a'-c)}{2}}, q^{\frac{B(a'+c)}{2}}, q^{Ba'}; q^{Ba'})_\infty}{(q^n; q^n)_\infty} \,.$$

We note that $\phi_{ka',B,kc,n}(q) = \phi_{a',kB,c,n}(q)$ and $\phi_{a',kB,c,kn}(q) = \phi_{a',B,c,n}(q^k)$. Therefore without loss of generality we assume $a' \perp c$ and $B \perp n$.

We write

$$\phi(q) = \sum_{j=0}^{\infty} \phi_j q^j.$$

Conjecture 3.4. *We have* $\phi_j \phi^{(i)}_{j+n} \geq 0$ *for all* $j \in \mathbb{Z}_{\geq 0}$.

If $n = 1$ then all factors in the numerator of $\phi(q)$ cancel with factors in the denominator and therefore all coefficients ϕ_j are clearly positive.

By Theorem 3.2, we can always write $\phi(q)$ as a sum of Virasoro characters (usually in several ways). This fact can be used to prove several cases of Conjecture 3.4.

Theorem 3.5. *Conjecture 3.4 holds in each of the following cases:*

(1) *all odd prime divisors of* n *divide* a'*;*
(2) n *is a prime number,* B *is odd.*

Theorem 3.5 is proved in Section 4.3. Theorem 3.5 in the case of $a' = 3, c = 1, B = 1$ and prime n is proved in [A1].

Remark 3.6. It follows immediately from Theorem 3.2 that coefficient ϕ_j is zero unless there exists a 2-contributing pair (r, s) such that $((p'rb - psb')^2 - (cB)^2)/(4Baa') - j$ is divisible by n. Equivalently, coefficient ϕ_j is zero unless there exists an integer m such that $mB(a'm + c)/2 - j$ is divisible by n, see Section 4.1.

Remark 3.7. The results similar to Theorem 3.5 also hold for the products appearing in the left hand sides of formulae (3.2) and (3.3).

3.2. **Quintuple products.** We set

$$a = 3.$$

Then p', a', b' are not divisible by 3, p is divisible by 3.

We call a pair of integers (r, s) *3-contributing of the first type* if

$$0 < r < p/b, \qquad 0 < s < p'/b', \qquad \frac{p'r/b' - ps/b - a' + 3c}{2aa'} \in \mathbb{Z}.$$

We call a pair of integers (r,s) *3-contributing of the second type*

$$0<r<p/b, \qquad 0<s<p'/b', \qquad \frac{p'r/b'+ps/b+a'-3c}{2aa'}\in\mathbb{Z}.$$

We denote the set of all 3-contributing pairs of type j by $\mathcal{A}_j^{(3)}$, $j=1,2$. We call pair of integers (r,s) *3-contributing* if (r,s) is either 3-contributing of the first type or 3-contributing of the second type. We denote the set of all 3-contributing pairs by $\mathcal{A}^{(3)}$.

Lemma 3.8. *We have* $\mathcal{A}_1^{(3)}\bigcap\mathcal{A}_2^{(3)}=\emptyset$.

If $0<r<r+2a<p/b$ *, then* $(r,s)\in\mathcal{A}_j^{(3)}$ *if and only if* $(r+2a,s)\in\mathcal{A}_j^{(3)}$.

If $0<s<s+2a'<p'/b'$ *, then* $(r,s)\in\mathcal{A}_j^{(3)}$ *if and only if* $(r,s+2a')\in\mathcal{A}_j^{(3)}$. □

Theorem 3.9. *We have the identity of formal power series in q:*

$$\frac{(q^{Bc},q^{B(2a'-c)},q^{2Ba'};q^{2Ba'})_\infty(q^{2B(a'+c)},q^{2B(a'-c)};q^{4Ba'})_\infty}{(q^n;q^n)_\infty}=$$
$$q^{\frac{(p-p')^2-(a'-3c)^2B^2}{4Baa'}}\left(\sum_{(r,s)\in\mathcal{A}_1^{(3)}}\chi^{(p,p')}_{rb,sb'}(q^n)-\sum_{(r,s)\in\mathcal{A}_2^{(3)}}\chi^{(p,p')}_{rb,sb'}(q^n)\right). \quad (3.4)$$

Theorem 3.9 is proved in Section 4.4. The cases $n=1,2$ of Theorem 3.9 can be found in [BF], see also [FFW].

We note that there are n summands on the left hand side of formula (3.4), moreover, $\chi^{(p,p')}_{rb,sb'}$ is present only if $\chi^{(p,p')}_{p-rb,p'-sb'}$ is not present, see Lemma 4.4.

If n is even then we have formulae which differ from (3.4) only by the choice of signs.

For a 3-contributing pair (r,s) define the integer $f_{r,s}$ as follows. If (r,s) is a 3-contributing pair of the first kind we set

$$f_{r,s}=\frac{p'r/b'-ps-a'+3c}{2aa'}.$$

If (r,s) is a 3-contributing pair of the second kind we set

$$f_{r,s}=\frac{p'r/b'+ps-a'+3c}{2aa'}.$$

Theorem 3.10. *Let p'/b' be even or let p/b be even and c odd. We have the identity of formal power series in q:*

$$\frac{(-q^{Bc},-q^{B(2a'-c)},q^{2Ba'};q^{2Ba'})_\infty(q^{2B(a'+c)},q^{2B(a'-c)};q^{4Ba'})_\infty}{(q^n;q^n)_\infty}=$$
$$q^{\frac{(p-p')^2-(a'-3c)^2B^2}{4Baa'}}\left(\sum_{(r,s)\in\mathcal{A}_1^{(3)}}(-1)^{f_{r,s}}\chi^{(p,p')}_{rb,sb'}(q^n)-\sum_{(r,s)\in\mathcal{A}_2^{(3)}}(-1)^{f_{r,s}}\chi^{(p,p')}_{rb,sb'}(q^n)\right). \quad (3.5)$$

Let p'/b' be divisible by 4 or let p/b and c be divisible by 4. We have the identity of formal power series in q:

$$\frac{(q^{Bc},-q^{B(2a'-c)},-q^{2Ba'};-q^{2Ba'})_\infty(-q^{2B(a'+c)},-q^{2B(a'-c)};q^{4Ba'})_\infty}{(q^n;q^n)_\infty}=$$

$$q^{\frac{(p-p')^2-(a'-3c)^2B^2}{4Baa'}}\left(\sum_{(r,s)\in\mathcal{A}_1^{(3)}}(-1)^{\frac{f_{r,s}(f_{r,s}-1)}{2}}\chi^{(p,p')}_{rb,sb'}(q^n)-\sum_{(r,s)\in\mathcal{A}_2^{(3)}}(-1)^{\frac{f_{r,s}(f_{r,s}-1)}{2}}\chi^{(p,p')}_{rb,sb'}(q^n)\right).\quad(3.6)$$

Let p'/b' be divisible by 4 or let p/b and $c+2$ be divisible by 4. We have the identity of formal power series in q:

$$\frac{(-q^{Bc},q^{B(2a'-c)},-q^{2Ba'};-q^{2Ba'})_\infty(-q^{2B(a'+c)},-q^{2B(a'-c)};q^{4Ba'})_\infty}{(q^n;q^n)_\infty}=$$

$$q^{\frac{(p-p')^2-(a'-3c)^2B^2}{4Baa'}}\left(\sum_{(r,s)\in\mathcal{A}_1^{(3)}}(-1)^{\frac{f_{r,s}(f_{r,s}+1)}{2}}\chi^{(p,p')}_{rb,sb'}(q^n)-\sum_{(r,s)\in\mathcal{A}_2^{(3)}}(-1)^{\frac{f_{r,s}(f_{r,s}+1)}{2}}\chi^{(p,p')}_{rb,sb'}(q^n)\right).\quad(3.7)$$

Theorem 3.10 is proved in Section 4.5.

We apply Theorem 3.9 to study the signs of the coefficients of products.

Fix natural numbers a',B,c,n such that $a'>c$, $a'\perp 3$. Define the formal power series $\psi(q)$ by the formula:

$$\psi(q)=\psi_{a',B,c,n}(q):=\frac{(q^{Bc},q^{B(2a'-c)},q^{2Ba'};q^{2Ba'})_\infty(q^{2B(a'+c)},q^{2B(a'-c)};q^{4Ba'})_\infty}{(q^n;q^n)_\infty}.$$

We note that $\psi_{ka',B,kc,n}(q)=\psi_{a',kB,c,n}(q)$ and $\psi_{a',kB,c,kn}(q)=\psi_{a',B,c,n}(q^k)$. Therefore without loss of generality we assume $a'\perp c$ and $B\perp n$.

We write

$$\psi(q)=\sum_{j=0}^{\infty}\psi_j q^j.$$

Conjecture 3.11. *We have $\psi_j\psi_{j+n}\geq 0$ for all $j\in\mathbb{Z}_{\geq 0}$.*

If $n=1$ then all factors in the numerator of $\psi(q)$ cancel with factors in the denominator and therefore all coefficients ψ_j are clearly positive. Thus Conjecture 3.4 is obviously true when $n=1$.

Note that by Theorem 3.2, we can always write $\psi(q)$ as a sum of Virasoro characters (usually in several ways). This fact can be used to prove some cases of Conjecture 3.11.

Theorem 3.12. *Conjecture 3.11 is true if all prime divisors of n different from 3 divide a'.*

Theorem 3.12 is proved in Section 4.6.

Remark 3.13. The results similar to Theorem 3.12 also hold for the products appearing in the left hand sides of formulae (3.5),(3.6) and (3.7).

Remark 3.14. It follows immediately from Theorem 3.9 that coefficient ψ_j is zero unless there exists a 3-contributing pair (r,s) such that $((p'rb-psb')^2-(a-3c)^2B^2)/(4Baa')-j$ is divisible by n. Equivalently, coefficient ψ_j is zero unless there exists an integer m such that $mB(3a'm+a'-3c)-j$ is divisible by n, see Section 4.4.

Remark 3.15. The products appearing in the right hand sides of our formulae satisfy some obvious relations. For example for odd a' and c, $a'>c$, we have

$$\phi_{a',1,c,1}(q)\prod_{j=1,\ j\neq(c+1)/2}^{(a'-1)/2}\phi_{a',1,2j-1,a'}(q)=1.$$

Theorems 3.2 and 3.9 can be used to replace $\phi_{a',B,c,n}$ and $\psi_{a',B,c,n}$ in this and similar formulae via alternating sums of Virasoro characters. That leads to identities which involve alternating sums of products of Virasoro characters.

4. Proofs

4.1. **Proof of Theorem 3.2.** The Jacobi triple product identity (see for example (2.2.10) in [A2]) reads:

$$(v,u,u^{-1}v;v)_\infty=\sum_{j\in\mathbb{Z}}(-1)^j u^j v^{j(j-1)/2}.$$

Substituting

$$v=q^{Ba'},\qquad u=q^{B(a'+c)/2}, \tag{4.1}$$

and changing the summation index j to $-j$ we obtain the following formula for the right hand side of (3.1):

$$\frac{(q^{B(a'-c)/2},q^{B(a'+c)/2},q^{Ba'};q^{Ba'})_\infty}{(q^n;q^n)_\infty}=\sum_{j\in\mathbb{Z}}\frac{(-1)^j q^{jB(a'j+c)/2}}{(q^n;q^n)_\infty}\ .$$

Substituting further $j=2nk+m$, where $k\in\mathbb{Z}$, $m\in\{0,\dots,2n-1\}$, we obtain:

$$\sum_{j\in\mathbb{Z}}\frac{(-1)^j q^{jB(a'j+c)/2}}{(q^n;q^n)_\infty}=\sum_{m=0}^{2n-1}(-1)^m q^{mB(a'm+c)/2}\sum_{k\in\mathbb{Z}}\frac{q^{nkB(2a'nk+2a'm+c)}}{(q^n;q^n)_\infty}\ . \tag{4.2}$$

After substituting Rocha-Caridi formula (2.1) for the Virasoro characters in the left hand side of formula (3.1), we obtain n positive and n negative terms of the form $q^{x_j}\sum_{k\in\mathbb{Z}}q^{nk(pp'k+y_j)}/(q^n;q^n)_\infty$ with some x_j,y_j. We claim that after a linear change of the summation index these terms match the $2n$ terms in the right hand side of (4.2).

Lemma 4.1. *The pair (r,s) is 2-contributing if and only if $0<r<p/b$, $0<s<p'/b'$, r is odd, $ps\equiv bc \pmod{a'}$.*

There are exactly n 2-contributing pairs.

If (r,s) is a 2-contributing pair then $(p/b-r,p'/b'-s)$ is not a 2-contributing pair.

If (r,s) is a 2-contributing pair then both $(p'r/b'-ps/b+c)$ and $(p'r/b'+ps/b-c)$ are divisible by $2a'$.

Proof. If (r,s) is a 2-contributing pair then we clearly have that r is odd and $ps/b\equiv c \pmod{a'}$. If r is odd and $ps/b\equiv c \pmod{a'}$, then clearly $(p'r/b'-pr/b+c)/(2a')$ and $(p'r/b'+pr/b-c)/(2a')$ are integers. The sum of these two integers equals to $p'r/(a'b')$ which is odd. Therefore exactly one of the numbers $(p'r/b'-pr/b+c)/(2aa')$ and $(p'r/b'+pr/b-c)/(2aa')$ is an integer and (r,s) is a 2-contributing pair.

Note that p and a' are relatively prime and therefore $p(ka'+1),p(ka'+2),\dots,p((k+1)a'-1)$ are all different and non-zero modulo a'. Therefore exactly one of these numbers has the same residue as bc modulo a'. It follows that we have $p'/(a'b')$ choices for s and similarly we have $p/(ab)$ independent choices for r. Thus we have $pp'/(aa'B)=n$ 2-contributing pairs.

If (r,s) is a 2-contributing pair then $ps/b\equiv c \pmod{a'}$ and therefore $p(p'/b'-s)/b\equiv -c \pmod{a'}$. Since a' is odd, c and $-c$ have different residues and the pair $(p/b-r,p'/b'-s)$ is not 2-contributing.

The numbers $p'r/b'\pm ps/b\pm c$ are even integers for all choices of pluses and minuses. Also $ps/b-c$ and $p'r/b'$ are both divisible by a'. Since a' is odd, the last statement of the lemma follows. ☐

For a 2-contributing pair (r,s), define integers $m_{r,s}$ and $\bar m_{r,s}$ as follows. Set $x_{r,s}=1$ if $p'r/b'-ps/b+c>0$ and $x_{r,s}=0$ if $p'r/b'-ps/b-c\le 0$. Then define

$$\begin{aligned} m_{r,s} &= 2nx_{r,s}-(p'r/b'-ps/b+c)/(2a'),\\ \bar m_{r,s} &= (p'r/b'+ps/b-c)/(2a'). \end{aligned} \tag{4.3}$$

We clearly have $0\le m_{r,s}\le 2n-1$, $0\le \bar m_{r,s}\le 2n-1$.

Lemma 4.2. *The $2n$ numbers $\{m_{r,s},\bar m_{r,s}\}$ are all distinct.*

Proof. If $m_{r_1,s_1}=m_{r_2,s_2}$ then

$$(p'r_1/b'-ps_1/b)-(p'r_2/b'-ps_2/b)=p'(r_1-r_2)/b'-p(s_1-s_2)/b$$

is divisible by $4a'n=2pp'/B$. The divisibility by p/b gives $r_1=r_2$ and the divisibility by p'/b' gives $s_1=s_2$.

If $\bar m_{r_1,s_1}=\bar m_{r_2,s_2}$ then

$$(p'r_1/b'+ps_1/b)-(p'r_2/b'+ps_2/b)=p'(r_1-r_2)/b'+p(s_1-s_2)/b$$

is zero and hence it is divisible by $2pp'/B$. Therefore $r_1=r_2$ and $s_1=s_2$.

If $m_{r_1,s_1}=\bar m_{r_2,s_2}$ then

$$(p'r_1/b'-ps_1/b)+(p'r_2/b'+ps_2/b)=p'(r_1+r_2)/b'-p(s_1-s_2)/b$$

is divisible by $2pp'/B$. The divisibility by p/b and by p'/b' implies $s_1 = s_2$ and $r_1+r_2 = p/b$. It leads to a conclusion that $2pp'/B$ divides pp'/B which is a contradiction. $\square$

Lemma 4.3. *We have*

$$(-1)^{m_{r,s}} = (-1)^{(p'r/b'-ps/b+c)/2}, \qquad \bar{m}_{r,s} = -(-1)^{(p'r/b'-ps/b+c)/2}.$$

In particular

$$\begin{aligned} (-1)^{m_{r,s}} = -(-1)^{\bar{m}_{r,s}} = 1 \qquad &\text{if} \quad (r,s) \in \mathcal{A}_1^{(2)}, \\ -(-1)^{m_{r,s}} = (-1)^{\bar{m}_{r,s}} = 1 \qquad &\text{if} \quad (r,s) \in \mathcal{A}_2^{(2)}. \end{aligned}$$

Proof. The first equation follows from the definition since a' is odd.

Since p/b is even and c is odd, we have

$$(-1)^{\bar{m}_{r,s}} = (-1)^{(p'r/b'+ps/b-c)/2} = (-1)^{(p'r/b'-ps/b+c)/2+(ps/b-c)} = -(-1)^{(p'r/b'-ps/b+c)/2}.$$

The rest of the lemma is obvious. $\square$

Finally, for a 2-contributing pair (r,s) we have

$$(-1)^{\frac{p'r/b'-ps/b+c}{2}} q^{\frac{(p-p')^2-(cB)^2}{8Ba'}} q^{-n\frac{(p-p')^2}{4pp'}} \sum_{k\in\mathbb{Z}} q^{\frac{n}{4pp'}(2pp'k+p'rb-psb')^2} =$$

$$(-1)^{m_{r,s}} \sum_{k\in\mathbb{Z}} q^{\frac{n}{4pp'}\left(\left(2pp'(-k-x_{r,s})+2pp'x_{r,s}-2a'Bm_{r,s}-cB\right)^2-(cB)^2\right)} =$$

$$= (-1)^{m_{r,s}} q^{m_{r,s}B(a'm_{r,s}+c)/2} \sum_{k\in\mathbb{Z}} q^{nkB(2a'nk+2a'm_{r,s}+c)}.$$

Similarly:

$$-(-1)^{\frac{p'r/b'-ps/b+c}{2}} q^{\frac{(p-p')^2-(cB)^2}{8Ba'}} q^{-n\frac{(p-p')^2}{4pp'}} \sum_{k\in\mathbb{Z}} q^{\frac{n}{4pp'}(2pp'k+p'rb+psb')^2} =$$

$$= (-1)^{\bar{m}_{r,s}} q^{\bar{m}_{r,s}B(a'\bar{m}_{r,s}+c)/2} \sum_{k\in\mathbb{Z}} q^{nkB(2a'nk+2a'\bar{m}_{r,s}+c)}.$$

Theorem 3.2 is proved.

4.2. **Proof of Theorem 3.3.** The proof of Theorem 3.3 is similar to the proof of Theorem 3.2. The only difference is in signs.

To prove formula (3.2), we change the substitution (4.1) to

$$v = -q^{Ba'}, \qquad u = -q^{B(a'+c)/2},$$

and observe that since n is even, a' is odd,

$$\frac{m_{r,s}(m_{r,s}-1)}{2} - \frac{\bar{m}_{r,s}(\bar{m}_{r,s}-1)}{2} = \frac{(m_{r,s}-\bar{m}_{r,s})(m_{r,s}+\bar{m}_{r,s}-1)}{2}$$

has the same parity as

$$\frac{(p'r/b')(ps/b-c-a')}{2}\,.$$

This number is odd because $a'+c$ is even but not divisible by 4, p/b is divisible by 4 and $p'r/b$ is odd. This observation replaces Lemma 4.3.

To prove formula (3.3), we change the substitution 4.1 to

$$v=-q^{Ba'},\qquad u=q^{B(a'+c)/2},$$

and observe that since n is even, a' is odd,

$$\frac{m_{r,s}(m_{r,s}+1)}{2}-\frac{\bar m_{r,s}(\bar m_{r,s}+1)}{2}=\frac{(m_{r,s}-\bar m_{r,s})(m_{r,s}+\bar m_{r,s}+1)}{2}$$

has the same parity as

$$\frac{(p'r/b')(ps/b-c+a')}{2}\,.$$

This number is odd because $a'-c$ is even but not divisible by 4, p/b is divisible by 4 and $p'r/b$ is odd. This observation replaces Lemma 4.3.

4.3. **Proof of Theorem 3.5.** Let all odd prime divisors of n divide a'.

Consider $2n$ numbers $\{Bm(a'm+c)/2,\ m=0,1,\dots,2n-1\}$. We claim that for each $j\in\{0,\dots,n-1\}$, exactly two of these $2n$ numbers have residue j modulo n.

Consider the following equation for $x\in\{0,1,\dots,2n-1\}$:

$$\frac{Bm_0(a'm_0+c)}{2}\equiv\frac{Bx(a'x+c)}{2}\pmod n. \tag{4.4}$$

To establish our claim, it is sufficient to show that for any $m_0\in\{0,1,\dots,2n-1\}$, equation (4.4) has exactly two solutions.

Since $B\perp n$, we cancel B on both sides and obtain that $(x-m_0)(a'(x+m_0)+c)/2$ is divisible by n. Write $n=2^dk$, where k is odd. Then from our assumptions we have $k\perp(a'(x+m_0)+c)$ and it follows that k divides $x-m_0$. Therefore x has the form $x=m_0+kl$ for some integer l satisfying $-m_0/k\le l<(2n-m_0)/k$.

If $x-m_0$ is even then $(a'(x+m_0)+c)$ is odd and it follows that $x-m_0$ is divisible by $2n$ and therefore $x=m_0$. If $x-m_0$ is odd then $(a'(x+m_0)+c)$ is divisible by 2^{d+1}. But since $a'k$ is odd, the 2^{d+1} numbers $\{a'((m_0+kl)+m_0)+c,\ -m_0/k\le l<(2n-m_0)/k\}$ all have different residues modulo 2^{d+1}. Therefore exactly one of them is divisible by 2^{d+1}.

Our claim is proved.

If $d>0$ then B is odd and we choose $p=2^{d+1}$. In such a case we have $b=1$, $b'=B$, $p'=ka'B$. If $d=0$ and $B=2^{\tilde d}\tilde k$ with odd $\tilde k$ then we choose $p=2^{\tilde d+1}$. In such a case we have $b=2^{\tilde d}$, $b'=\tilde k$, $p'=\tilde k a'n$.

Use Theorem 3.2 to write $\phi(q)$ as a sum of n Virasoro characters. It follows that for each $j\in\mathbb{Z}_{\ge0}$ we have exactly one term in the left hand side of (3.1) which contains q^j and moreover, this term is the same for j and $j+n$. Indeed, each Virasoro character

corresponds to two terms in (4.2) and as we have shown, exactly two terms in (4.2) contribute to q^j with a given j modulo n.

The first statement of the theorem is proved.

Let now n be an odd prime number and let B be odd. (Cf. [A1], proof of Theorem 1.)

Choose $p=2$. We have $p'=a'nB$, $b=1$, $b'=B$.

Consider equation (4.4). We claim that there are at most 4 solutions. Indeed $(x-m_0)(a'x+a'm_0+c)$ is divisible by n. Since n is prime then either $x-m_0$ or $a'x+a'm_0+c$ is divisible by n. In each case we obtain at most two values of x.

Therefore for each $j\in\mathbb{Z}_{\geq 0}$ we have at most two terms in the left hand side of (3.1) which contain q^j. But the difference of two $(2,p')$ Virasoro characters is known to have all coefficients of the same sign. It follows for example from the fermionic representation of $(2,p')$ characters used in the Rogers-Ramanujan-Gordon-Andrews identities, (see (7.3.7) in [A2]).

4.4. **Proof of Theorem 3.9.** The proof of Theorem 3.9 is similar to that of Theorem 3.2. The main difference is the use of the quintuple product identity as opposed to the triple product identity.

The quintuple product identity (see [Wt]) reads:

$$(v,u,u^{-1}v;v)_\infty(u^2v,u^{-2}v;v^2)_\infty=\sum_{j\in\mathbb{Z}}(u^{-3j}-u^{3j+1})v^{j(3j+1)/2}.$$

Substituting

$$v=q^{2Ba'},\qquad u=q^{Bc}, \tag{4.5}$$

we obtain the following formula for the right hand side of (3.4):

$$\frac{(q^{Bc},q^{B(2a'-c)},q^{2Ba'};q^{2Ba'})_\infty(q^{2B(a'+c)},q^{2B(a'-c)};q^{4Ba'})_\infty}{(q^n;q^n)_\infty}=\\ \sum_{j\in\mathbb{Z}}\frac{q^{jB(3a'j+a'-3c)}-q^{jB(3a'j+a'+3c)+Bc}}{(q^n;q^n)_\infty}.$$

Substituting further $j=nk+m$, where $k\in\mathbb{Z}$ and $m\in\{0,1,\dots,n-1\}$, we obtain:

$$\sum_{j\in\mathbb{Z}}\frac{q^{jB(3a'j+a'-3c)}-q^{jB(3a'j+a'+3c)+Bc}}{(q^n;q^n)_\infty}=\sum_{m=0}^{n-1}\Big(q^{mB(3a'm+a'-3c)}\sum_{k\in\mathbb{Z}}\frac{q^{nkB(3a'nk+6a'm+a'-3c)}}{(q^n;q^n)_\infty}-\\ -q^{mB(3a'm+a'+3c)+Bc}\sum_{k\in\mathbb{Z}}\frac{q^{nkB(3a'nk+6a'm+a'+3c)}}{(q^n;q^n)_\infty}\Big)\,. \tag{4.6}$$

After substituting the Rocha-Caridi formula (2.1) for the Virasoro characters in the left hand side of formula (3.4), we obtain n positive and n negative terms of the form $q^{x_j}\sum_{k\in\mathbb{Z}}q^{nk(pp'k+y_j)}/(q^n;q^n)_\infty$ for some x_j,y_j. We claim that after a linear change of the

summation index these terms match the n positive and n negative terms in the right hand side of (4.6).

Lemma 4.4. *There are exactly n 3-contributing pairs.*

If (r,s) is a 3-contributing pair then $(p/b-r, p'/b'-s)$ is not a 3-contributing pair.

The pair (r,s) is a 3-contributing pair of type 1 if and only if $p'r/b'+ps/b-a'-3c$ is divisible by $6a'$.

The pair (r,s) is a 3-contributing pair of type 2 if and only if $p'r/b'-ps/b+a'+3c$ is divisible by $6a'$.

Proof. First, consider the case when $p'/(a'b')$ is odd.

Then we claim that for any nonnegative integers k_1, k_2 such that $3k_1 < p/b$ and $a'k_2 < p'/b'$ there is exactly one 3-contributing pair (r,s) such that $3k_1 \le r < 3k_1+3$ and $a'k_2 \le s < a'k_2 + a'$.

Indeed there is exactly one pair (r_1, s_1) such that $3k_1 \le r_1 < 3k_1+3$, $a'k_2 \le s_1 < a'k_2+a'$ and $p'r_1/b' - ps_1/b - a' + 3c$ is divisible by $3a'$. The numbers r_1, s_1 are unique solutions (in the specified range) of equations $p'r_1/b' \equiv a' \pmod 3$ and $ps_1/b \equiv 3c \pmod{a'}$.

Similarly there is exactly one pair (r_2, s_2) such that $3k_1 \le r_2 < 3k_1+3$, $a'k_2 \le s_2 < a'k_2 + a'$ and $p'r_2/b' + ps_2/b + a' - 3c$ is divisible by $3a'$.

We have $s_1 = s_2$. Since $a' \perp 3$ we also have, $r_1 \ne r_2$, $r_1 \ne 3k_1$, $r_2 \ne 3k_1$ and therefore $|r_1 - r_2| = 1$. Recall that $p'/(a'b')$ is odd. It follows that exactly one of the two numbers $(p'r_2/b' - ps_2/b - a' + 3c)/(3a')$ and $(p'r_1/b' + ps_1/b + a' - 3c)/(3a')$ is even and we have exactly one 3-contributing pair.

Now, let $p'/(a'b')$ be even. Then we repeat the same argument. However, in this case, the numbers $(p'r_2/b' - ps_2/b - a' + 3c)/(3a')$ and $(p'r_1/b' + ps_1/b + a' - 3c)/(3a')$ have the same parity. But this parity is changed when k_2 is replaced by k_2+1. Therefore for half of the possible values of k_2 we have two 3-contributing pairs and there are no contributing pairs for the other half.

Let (r,s) be a 3-contributing pair of the first type, that is $p'r/b' - ps/b - a' + 3c$ is divisible by $6a'$. Then $p'r/b' - ps/b + a' - 3c$ and $p'r_1/b' + ps_1/b - a' + 3c$ are not divisible by $6a'$. The number $2pp'/B$ is divisible by $6a'$. It follows that $(p/b - r, p'/b' - s)$ is not a 3-contributing pair.

The case of a 3-contributing pair of the second type is done similarly.

If (r,s) is a 3-contributing pair of the first type then $2p'r/b' - 2a'$ is divisible by 3. In addition it is clearly divisible by $2a'$ and therefore it is divisible by $6a'$.

Similarly, if (r,s) is a 3-contributing pair of the second type then $2p'r/b' + 2a'$ is divisible by $6a'$.

The last two statements of the lemma follow. $\square$

If (r, s) is a 3-contributing pair of type 1, we define integers $x_{r,s}, \bar{x}_{r,s}, m_{r,s}, \bar{m}_{r,s}$ by the equality

$$\begin{aligned}\frac{p'r/b' - ps/b - a' + 3c}{6a'} &= nx_{r,s} + m_{r,s},\\ \frac{p'r/b' + ps/b - a' - 3c}{6a'} &= n\bar{x}_{r,s} + \bar{m}_{r,s},\end{aligned}$$

and the requirement $0 \leq m_{r,s} < n$, $0 \leq \bar{m}_{r,s} < n$.

If (r, s) is a 3-contributing pair of type 2, we define integers $x_{r,s}, \bar{x}_{r,s}, m_{r,s}, \bar{m}_{r,s}$ by the equality

$$\begin{aligned}-\frac{p'r/b' + ps/b + a' - 3c}{6a'} &= nx_{r,s} + m_{r,s},\\ -\frac{p'r/b' - ps/b + a' + 3c}{6a'} &= n\bar{x}_{r,s} + \bar{m}_{r,s},\end{aligned}$$

and the requirement $0 \leq m_{r,s} < n$, $0 \leq \bar{m}_{r,s} < n$.

Lemma 4.5. *The n numbers $\{m_{r,s}\}$ are all distinct. The n numbers $\{\bar{m}_{r,s}\}$ are also all distinct.*

Proof. If $m_{r_1,s_1} = m_{r_2,s_2}$ then

$$(p'r_1/b' - ps_1/b) - (p'r_2/b' - ps_2/b) = p'(r_1 - r_2)/b' - p(s_1 - s_2)/b$$

is divisible by $2pp'/B$ or

$$(p'r_1/b' - ps_1/b) + (p'r_2/b' + ps_2/b) = p'(r_1 + r_2)/b' - p(s_1 - s_2)/b$$

is divisible by $2pp'/B$.

In the former case the divisibility by p/b gives $r_1 = r_2$ and the divisibility by p'/b' gives $s_1 = s_2$. In the later case we similarly obtain $s_1 = s_2$ and $r_1 + r_2 = p/b$. It leads to a conclusion that $2pp'/B$ divides pp'/B which is a contradiction.

The case $\bar{m}_{r_1,s_1} = \bar{m}_{r_2,s_2}$ is done similarly. □

Finally, for a 3-contributing pair (r, s) of type 1 we have

$$q^{\frac{(p-p')^2 - B^2(a'-3c)^2}{12Ba'}} q^{-n\frac{(p-p')^2}{4pp'}} \sum_{k\in\mathbb{Z}} q^{\frac{n}{4pp'}(2pp'k + p'rb - psb')^2} =$$

$$\sum_{k\in\mathbb{Z}} q^{\frac{n}{4pp'}\left(\left(2pp'(k-\bar{x}_{r,s}) + 2pp'x_{r,s} + 6a'Bm_{r,s} + (a'-3c)B\right)^2 - B^2(a'-3c)^2\right)} =$$

$$= q^{m_{r,s}B(3a'm_{r,s} + a' - 3c)} \sum_{k\in\mathbb{Z}} q^{nkB(3a'nk + 6a'm_{r,s} + a' - 3c)}.$$

Similarly:

$$q^{\frac{(p-p')^2-B^2(a'-3c)^2}{12Ba'}}q^{-n\frac{(p-p')^2}{4pp'}}\sum_{k\in\mathbb{Z}}q^{\frac{n}{4pp'}(2pp'k+p'rb+psb')^2}=$$

$$\sum_{k\in\mathbb{Z}}q^{\frac{n}{4pp'}\left(\left(2pp'(k-x_{r,s})+2pp'x_{r,s}+6a'B\bar{m}_{r,s}+(a'+3c)B\right)^2-B^2(a'-3c)^2\right)}=$$

$$=q^{\bar{m}_{r,s}B(3a'\bar{m}_{r,s}+a'+3c)+Bc}\sum_{k\in\mathbb{Z}}q^{nkB(3a'nk+6a'\bar{m}_{r,s}+a'+3c)}.$$

The computation for a 3-contributing pair of type 2 is similar. Theorem 3.9 is proved.

4.5. **Proof of Theorem 3.10.** The proof of Theorem 3.10 is similar to the proof of Theorem 3.9. The only difference is in signs.

To prove formula (3.5), we change the substitution (4.5) to

$$v=q^{2Ba'},\qquad u=-q^{Bc},$$

and observe that $m_{r,s}+\bar{m}_{r,s}$ has the same parity as $(p'r/b'-a')/a'$. This number is clearly odd if $p'/(a'b')$ is even. If $p'/(a'b')$ is odd and n is even then p/b is even, a' is odd and if c is also odd then r is even and therefore $(p'r/b'-a')/a'$ is odd.

To prove formula (3.6), we change the substitution (4.5) to

$$v=-q^{2Ba'},\qquad u=q^{Bc},$$

and observe that

$$\frac{m_{r,s}(3m_{r,s}+1)}{2}-\frac{\bar{m}_{r,s}(3\bar{m}_{r,s}+1)}{2}=\frac{(m_{r,s}-\bar{m}_{r,s})(3m_{r,s}+3\bar{m}_{r,s}+1)}{2}$$

has the same parity as

$$\frac{1}{2}\,\frac{p'r}{a'b'}\,\frac{(ps/b-3c)}{a'}\,.$$

This number is even. Indeed, n is divisible by 4, hence if $p'/(a'b')$ is even then $p'/(a'b')$ is divisible by 4, and if $p'/(a'b')$ is odd then p/b is divisible by 4.

To prove formula (3.7), we change the substitution (4.5) to

$$v=-q^{2Ba'},\qquad u=-q^{Bc},$$

and observe that

$$\frac{m_{r,s}(3m_{r,s}-5)}{2}-\frac{\bar{m}_{r,s}(3\bar{m}_{r,s}-5)}{2}=\frac{(m_{r,s}-\bar{m}_{r,s})(3m_{r,s}+3\bar{m}_{r,s}-5)}{2}$$

has the same parity as

$$\frac{1}{2}\,\frac{(3p'r-2a'b')}{a'b'}\,\frac{(ps/b-3c)}{a'}\,.$$

This number is odd. Indeed if p/b is divisible by 4 and c is even then $p'r$ is odd, and if $p'/(a'b')$ is divisible by 4 then $(ps/b-3c)/a'$ is odd.

4.6. **Proof of Theorem 3.12.** The proof of Theorem 3.12 is similar to that of Theorem 3.5.

Namely, we write $n = 3^d k$ where $k \perp 3$ and show that the n numbers $Bm(3a'm+a'-3c)$, $m = 0, \dots, n-1$, are all different modulo n. It follows that the n terms of the left hand side of (3.4) all contribute to different coefficients and therefore there is no further subtraction.

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